

Dimensional Analysis: A Centenary Update

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Abstract

It is time to renew old ways of thinking about dimensional analysis. Specifically, more than $n - r$ invariants and more than one functional relation between invariants need to be considered simultaneously. Thus generalized, dimensional analysis can yield more information than previously recognized.

Buckingham's Π theorem [Bu14] and its predecessors [Va92, Ria11] were formulated 100 years or more ago. The basic principles of dimensional analysis have remained unchanged since then. Yet, a careful investigation reveals that dimensional analysis rests on presuppositions which unnecessarily limit the scope and power of the analysis. This is implied by the examination of basic notions related to dimensional analysis in [Jo14]; this article will explain more explicitly how dimensional analysis should be extended to transcend its self-imposed limitations.

1 Dimensional analysis should be liberated from some traditional constraints

Dimensional analysis as depicted in well-known classical expositions [Bri22, Se93, Ba93], can be summarized as follows. We want to express a dependent quantity q as a function of independent quantities q_1, \dots, q_{n-1} , where $n > 1$; thus, we assume that $q = \Psi(q_1, \dots, q_{n-1})$, or equivalently and sometimes more conveniently,

$$q_1 = \Psi(q_2, \dots, q_n). \quad (1)$$

We also assume that the quantities q_1, \dots, q_n can all be expressed in terms of one or more fundamental units of measurement, corresponding to m dimensions such as Length, Time and Mass, where $1 \leq m < n$. Dimensional analysis makes it possible to find $n - r$ so-called π -groups π_1, \dots, π_{n-r} such that (1) can be written as

$$\pi_1 = \Phi(\pi_2, \dots, \pi_{n-r}), \quad (2)$$

for some r such that $0 \leq r \leq m$. The π -groups are invariant products of powers of q_1, \dots, q_n , meaning that the numerical values of these products do not depend

on the fundamental units of measurement used to express q_1, \dots, q_n . Relations of the form (2) can be rewritten as

$$q_1^c = \prod_{j=1}^r q_{i_j}^{c_j} \Phi(\pi_1, \dots, \pi_{n-r}) \quad \text{or} \quad q_1 = \prod_{j=1}^r q_{i_j}^{c_j/c} \Phi(\pi_1, \dots, \pi_{n-r})^{1/c},$$

where c, c_j are integers and q_1, q_{i_j} are distinct quantities.

In many contemporary expositions of dimensional analysis, the way in which q_1, \dots, q_n are expressed in terms of fundamental units of measurement or corresponding dimensions is summarized in a *dimensional matrix* $[a_{ij}]$, where a_{ij} is the dimensionality of q_j relative to the dimension D_i , meaning that $[q_j] = D_1^{a_{1j}} \dots D_i^{a_{ij}} \dots D_m^{a_{mj}}$ [Jo14]. The invariants (π -groups) are obtained from this matrix. It is not difficult to show that r is equal to the rank of the dimensional matrix. The introduction of notions and techniques from linear algebra does not change the general way of thinking about dimensional analysis, however.

Dimensional analysis as described above has two limitations:

- (P1) *Not more than one relation of the form (2) is considered.*
- (P2) *Not more than $n - r$ invariants are considered.*

The main purpose of this article is to present a generalized form of dimensional analysis which is not constrained by these limitations. Before proceeding, it should be noted, though, that there exists a tension between the traditional, abstract description of dimensional analysis and the method used in practice. This method revolves around the notion of *repeating variables* (specifically, repeating quantities), and is based on the observation that every invariant in a relation of the form (2) can be written as

$$\frac{p^c}{p_1^{c_1} \dots p_r^{c_r}} \quad \text{or} \quad \frac{p}{p_1^{c_1/c} \dots p_r^{c_r/c}},$$

where p, p_i are distinct quantities, c, c_i are integers. Here, each p which occurs in a numerator occurs in exactly one invariant in the relation, while each p_i which occurs in a denominator may occur in all invariants in the relation – hence, repeating variables – but *the set of quantities occurring in the denominators can in general be chosen in more than one way.*

As we shall see, different sets of repeating variables give different relations of the form (2). Hence, there may be more than one such relation, and as the $n - r$ invariants in one relation are not the same as the $n - r$ invariants in another relation, the total number of invariants may be greater than $n - r$. The emphasized fact is thus an anomaly from the point of view of dimensional analysis in the tradition of the Π theorem. Simplifying and idealizing, one can say that, apart from quietly ignoring the anomaly, three ways of dealing with it are found in the literature:

1. Some authors try to re-establish uniqueness by suggesting criteria for choosing the 'right' repeating variables, although they may not insist that finding one 'right' set of repeating variables is always possible.

2. Some authors acknowledge non-uniqueness but downplay it, arguing that essentially the same result is obtained no matter which formally well-behaved set of repeating variables is chosen.
3. Some authors accept non-uniqueness in practice, thus effectively abandoning (P1) and (P2), but pay lip service to the traditional formulation of dimensional analysis, thereby disconnecting theory and practice.

These positions will be described more fully below, relating each one to the approach proposed here.

2 Invariants versus active invariants

An example, adapted from Sedov [Se93], will be used to introduce the new way of thinking about dimensional analysis. Consider a fluid flowing through a cylindrical pipe; let $\Delta P/\ell$ be the pressure drop per unit length, ρ the density of the fluid, μ the viscosity of the fluid, d the diameter of the pipe, and u the (mean) velocity of the fluid. The dimensional matrix for these quantities is

$$\begin{array}{c} L \\ T \\ M \end{array} \left\| \begin{array}{ccccc} \Delta P/\ell & \rho & \mu & d & u \\ -2 & -3 & -1 & 1 & 1 \\ -2 & 0 & -1 & 0 & -1 \\ 1 & 1 & 1 & 0 & 0 \end{array} \right\|.$$

The rank of this matrix is 3. There is a well-known invariant involving the quantities ρ, μ, d, u , namely the Reynolds number

$$\frac{\rho d u}{\mu} = \text{Re} = \Pi_1.$$

Sedov also considers another invariant

$$\frac{(\Delta P/\ell) d}{\rho u^2} = \Pi.$$

and we now have $n - r = 5 - 3 = 2$ invariants. Sedov asserts that Re is the only possible invariant involving ρ, μ, d and u , which means that any invariant which depends on ρ, μ, d, u actually depends on $\rho d u \mu^{-1}$. Thus, assuming that $\Delta P/\ell = \Psi(\rho, \mu, d, u)$, we have $\Pi = \Phi(\Pi_1)$, or

$$\Delta P/\ell = \rho u^2 d^{-1} \Phi(\text{Re}).$$

However, we have not looked for invariants systematically. We stopped after finding two invariants, guided by the traditional way of thinking about dimensional analysis. But there is actually no ground for assuming that these are the only invariants for this dimensional matrix, or even the only invariants of interest, so let us see what happens if we look for more invariants.

In a sense, one should not look for all invariants, however. To begin with, if π is an invariant then π^k is also an invariant for any non-zero integer k ; yet,

all these invariants are essentially the same. To better understand what this means, consider a product of powers of quantities $q_{i_1}^{c_1} \dots q_{i_k}^{c_k}$, and let \mathbf{v}_j be the (column) vector corresponding to q_{i_j} in the dimensional matrix. As explained in Section 3, $q_{i_1}^{c_1} \dots q_{i_k}^{c_k}$ is an invariant if and only if the equation with integer coefficients

$$c_1 \mathbf{v}_1 + \dots + c_k \mathbf{v}_k = \mathbf{0}$$

holds. It is clear that this equation still holds, and still has integer coefficients, if the integers c_1, \dots, c_k are multiplied by a non-zero integer, or divided by a common divisor of c_1, \dots, c_k . Thus, we can represent all such equations by an equation where c_1, \dots, c_k are relatively prime. There are exactly two such equations, which can be obtained from each other through multiplying c_1, \dots, c_k by -1 . The corresponding set of invariants can then be represented by *pairs of minimal invariants*, each of which can be obtained from the other by a sign flip. Expressed otherwise, there are sets of equivalent invariants such that each set can be represented – uniquely up to a sign flip – by an invariant $q_{i_1}^{c_1} \dots q_{i_k}^{c_k}$ such that c_1, \dots, c_k are relatively prime.

The two invariants considered by Sedov could thus be regarded as a set of pairs of minimal invariants

$$\left\{ \left(\frac{\rho du}{\mu} \right)^{\pm 1}, \left(\frac{(\Delta P/\ell) d}{\rho u^2} \right)^{\pm 1} \right\}.$$

In simplified notation, this set can be denoted

$$\left\{ \frac{\rho du}{\mu}, \frac{(\Delta P/\ell) d}{\rho u^2} \right\}^{\pm 1}.$$

So far, we have only succeeded in reformulating the original question “Are there more than $n - r$ invariants?” into the more sophisticated question “Are there more than $n - r$ pairs of minimal invariants?”, however. The situation will be clarified in Section 3; a more informal discussion will suffice here.

Recall that the dimensional matrix given has rank 3, and let us accept that the sets of possible repeating quantities that we need to consider are $\{\rho, \mu, d\}$, $\{\rho, \mu, u\}$, $\{\rho, d, u\}$ and $\{\mu, d, u\}$. The corresponding homogeneous systems of linear equations have minimal integer solutions defining the following invariants:

$$\left\{ \frac{\Delta P/\ell}{\rho^{-1} \mu^2 d^{-3}}, \frac{\Delta P/\ell}{\rho^2 \mu^{-1} u^3}, \frac{\Delta P/\ell}{\rho d^{-1} u^2}, \frac{\Delta P/\ell}{\mu d^{-2} u}, \frac{\rho u d}{\mu} \right\}^{\pm 1}.$$

We can choose one invariant in each pair without loss of generality. Accordingly, dimensional analysis produces four – not one – possible ways of writing the relation $\Delta P/\ell = \Psi(\rho, \mu, d, u)$, and we can take these representations to be:

$$\begin{cases} \Delta P/\ell = \rho^{-1} \mu^2 d^{-3} \phi_1(\rho u d \mu^{-1}) & \text{(a)} \\ \Delta P/\ell = \rho^2 \mu^{-1} u^3 \phi_2(\rho u d \mu^{-1}) & \text{(b)} \\ \Delta P/\ell = \rho d^{-1} u^2 \phi_3(\rho u d \mu^{-1}) & \text{(c)} \\ \Delta P/\ell = \mu d^{-2} u \phi_4(\rho u d \mu^{-1}) & \text{(d)} \end{cases}.$$

As we have seen, Sedov gives formula (c), and other authors seem to have followed in his footsteps [Ba93].

When the internal roughness of the pipe can be disregarded as assumed above, (c) is equivalent to the Darcy-Weisbach equation, which can be written in notation similar to that used above as

$$\Delta P = f \frac{\ell}{d} \frac{\rho u^2}{2}.$$

Here, $f = f\left(\frac{\rho u d}{\mu}, \frac{\epsilon}{d}\right)$ is a “dimensionless” quantity and ϵ is a quantity of dimension L expressing the roughness of the pipe, with

$$f\left(\frac{\rho u d}{\mu}, \frac{\epsilon}{d}\right) = 2\Phi_3\left(\frac{\rho u d}{\mu}\right)$$

for the idealized case $\frac{\epsilon}{d} = 0$.

Is (c) the ‘right’ formula, then? Can we reject the three other formulas? No, no formula is ‘wrong’. All four formulas are equally correct and in fact interchangeable for the somewhat surprising reason that all contain less information than they seem to do at first sight. A closer look at formula (c), for example, reveals that it is not possible to conclude that $\Delta P/\ell$ is proportional to ρ and u^2 and inversely proportional to d , because ρ , μ and d also appear as arguments of Φ_3 . Corresponding conclusions apply to formulas (a), (b) and (d). The four formulas are different because the effect of *different* quantities are “hidden inside” the functions Φ_1 through Φ_4 . Specifically, the effects of u , d , μ and ρ are hidden in formulas (a), (b), (c) and (d), respectively. Let us consider two more cases to clarify the situation.

For Reynolds numbers less than a critical value Re_c , the flow through the pipe is laminar, which means that the flow is non-accelerated. This implies that the flow is not affected by ρ [Se93]. (Simply stated, acceleration depends on inertia, which depends on mass, which depends on density.) Thus, we obtain the following dimensional matrix for laminar flow through a pipe:

$$\begin{array}{c} L \\ T \\ M \end{array} \left\| \begin{array}{cccc} \Delta P/\ell & \mu & d & u \\ -2 & -1 & 1 & 1 \\ -2 & -1 & 0 & -1 \\ 1 & 1 & 0 & 0 \end{array} \right\|.$$

In this case, we need to consider only one pair of minimal invariants,

$$\left(\frac{\Delta P/\ell d^2}{\mu u}\right)^{\pm 1},$$

and the relation $\Delta P/\ell = \Psi(\mu, d, u)$ can be written in the form

$$\Delta P/\ell = \frac{\mu u}{d^2} \Phi() = K \frac{\mu u}{d^2}, \quad (3)$$

where K is a scalar constant.

This is obviously a special case of (d). It is worth noting that the Hagen–Poiseuille equation, which can be written in notation similar to that used here as

$$\Delta P = C \frac{\ell \mu Q}{d^4},$$

where C is a scalar constant and Q the volumetric flow rate, is equivalent to (3), since $Q = u (\pi/4) d^2$. Hence, this well-known equation can be regarded as a special case of (d).

Using the fact that

$$K \frac{\mu u}{d^2} = \Delta P / \ell = \frac{\rho u^2}{d} \Phi_3 \left(\frac{\rho u d}{\mu} \right)$$

for laminar flow, we can determine Φ_3 for corresponding values of Re , and we obtain $\Phi_3(x) = K/x$ for $x < \text{Re}_c$. The remaining three functions Φ_i can be derived in the same way, and we have

$$\begin{cases} \Phi_1(x) = Kx \\ \Phi_2(x) = K/x^2 \\ \Phi_3(x) = K/x \\ \Phi_4(x) = K \end{cases}$$

for laminar flow.

A case of particular historical interest will also be mentioned. In his article from 1883, "An experimental investigation of the circumstances which determine whether the motion of water shall be direct or sinuous, and of the law of resistance in parallel channels", Reynolds [Re83] not only discussed the critical Reynolds number (as it came to be called), but also stated a law concerning the pressure drop connected with the flow of water through a cylindrical pipe. In notation similar to that used here, his law is

$$\Delta P / \ell \frac{d^3}{\mu_\theta^2} = \Phi \left(\frac{du}{\mu_\theta} \right),$$

where μ_θ is the viscosity of water at temperature θ (p. 973). Considering that the only fluid used by Reynolds was water, and that the temperature of water affects its viscosity much more than its density, so that ρ can be regarded as a constant, Reynolds formula is obviously a special case of (a).

We have shown, then, that two of the representations of the relation $\Delta P / \ell = \Psi(\rho, \mu, d, u)$ correspond to well-known formulas in fluid dynamics, while one representation, although known to Reynolds, seems to have been forgotten, and one representation seems to have escaped notice altogether.

To return to the main theme, we conclude that in the interest of clarity a distinction should be made between *invariants* and *active invariants*. For example, in the main example we consider five invariants – or pairs of minimal invariants – but there are only two active invariants in each of the four relations derived. It is clear why this distinction is obscured in traditional dimensional analysis – it is not relevant when only one relation is considered.

Remark

Some textbooks treating dimensional analysis suggest criteria for choosing repeating quantities in order to help students to choose the 'right' set of repeating quantities, usually mixing 'formal' and 'substantial' criteria [Ce04]. 'Formal' criteria are criteria such as "Do not assign two quantities with the same dimensions to the same set of repeating quantities". Such criteria ensure that the mathematical assumptions underlying dimensional analysis are satisfied; specifically, they guarantee that a set of repeating quantities is a maximal set of independent quantities and does not contain the dependent quantity. 'Substantial' criteria are criteria such as "If possible, choose a simple quantity such as a length or a time as a repeating quantity". These criteria are meant to help students choose the 'right' groups of repeating quantities among the groups which satisfy the 'formal' criteria, so that, for example, one of the relations (a) – (d) is designated as the 'right' one. The question raised here is if criteria of the second kind are necessary or even useful.

3 Finding all invariants

Consider a dimensional matrix with integer coefficients:

$$\begin{matrix} D_1 \\ \dots \\ D_m \end{matrix} \left\| \begin{array}{ccc} q_1 & \dots & q_n \\ a_{11} & \dots & a_{1n} \\ \dots & & \dots \\ a_{m1} & \dots & a_{mn} \end{array} \right\| .$$

Recall [Jo14] that finding an invariant product $q_1^{c_1} \dots q_n^{c_n}$ of q_1, \dots, q_n is equivalent to finding a vector $(c_1, \dots, c_n) \in \mathbb{Z}^n$ such that

$$c_1 \mathbf{v}_1 + \dots + c_n \mathbf{v}_n = \mathbf{0}, \quad (4)$$

where \mathbf{v}_i is the column vector $[a_{1i}, \dots, a_{mi}]^T$ corresponding to q_i and $\mathbf{0}$ is the column vector with n zeros.

Let \mathbf{D} be the matrix whose columns are the column vectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, let r be the rank of \mathbf{D} , and assume without loss of generality that the last r columns of \mathbf{D} are independent as column vectors. Recall from linear algebra that the solution space for (4), $\mathbb{R}_0^n(\mathbf{D}) = \{(c_1, \dots, c_n) \mid c_i \in \mathbb{R}, \sum_i c_i \mathbf{v}_i = \mathbf{0}\}$, has dimension $n - r$, and has a basis of $n - r$ vectors of the forms

$$(1, 0, \dots, 0, b_{11}, \dots, b_{1r}), \dots, (0, \dots, 0, 1, b_{(n-r)1}, \dots, b_{(n-r)r}). \quad (5)$$

Thus, every solution of the equation system is a unique linear combination of these n -tuples. As \mathbf{D} is an integer matrix, all b_{ij} are ratios of integers. Let $b_i > 0$ be the lowest common denominator of b_{i1}, \dots, b_{ir} . Then,

$$I = \{(b_1, 0, \dots, 0, c_{11}, \dots, c_{1r}), \dots, (0, \dots, 0, b_{n-r}, c_{(n-r)1}, \dots, c_{(n-r)r})\},$$

where $c_{ij} = b_i b_{ij}$, is a basis for $\mathbb{R}_0^n(\mathbf{D})$ with only integer entries, and any integer solution of (4) is a unique linear combination with integer coefficients of elements of I .

This means that every invariant corresponding to a solution of (4) is a product of powers of $n - r$ invariants

$$q_1^{b_1} q_2^0 \dots q_{n-r}^0 q_{n-r+1}^{c_{11}} \dots q_n^{c_{1r}}, \dots, q_1^0 \dots q_{n-r-1}^0 q_{n-r}^{b_1} q_{n-r+1}^{c_{(n-r)1}} \dots q_n^{c_{(n-r)r}}.$$

Since we can disregard factors of the form q^0 [Jo14], these invariants can be written in the form

$$q_i^{b_i} \prod_{j=1}^r q_{n-r+j}^{c_{ij}} \quad (i = 1, \dots, n - r),$$

where $b_i > 0$, and each invariant can be reduced further to an invariant of the form

$$q_i^{b_i} \prod_{k=1}^{r_i} q_{n-r+j_k}^{c_{ij_k}} \quad (i = 1, \dots, n - r, \quad 0 \leq r_i \leq r), \quad (6)$$

where $b_i > 0$, $c_{ij_k} \neq 0$ for all i, j_k , and b_i, c_{ij_k} are relatively prime.

For each such invariant, the column vectors corresponding to quantities with non-zero exponents make up a set of non-independent column vectors. Such a set is even a minimal set of non-independent column vectors; the column vectors in any subset are independent because otherwise all exponents b_i, c_{ij_k} would not be non-zero.

Thus, for (a) every maximal (possibly empty) set of independent quantities or column vectors, there is (b) a (non-empty) set of (non-empty) minimal sets of non-independent quantities or column vectors, corresponding to invariants of the form (6). With terminology inspired by matroid theory, we can call a maximal set of independent quantities or column vectors a *basis set*, a minimal set of non-independent quantities or column vectors a *circuit set*, and we can rephrase the last sentence by saying that for any basis set given by the dimensional matrix there are one or more corresponding circuit sets. The set of circuit sets is in one-to-one correspondence with (c) a set of integer tuples (c_1, \dots, c_n) called *circuit tuples* and (d) a set of invariants $q_1^{c_1} \dots q_n^{c_n}$ called *circuit invariants*, where c_1, \dots, c_n are relatively prime, or, equivalently, (d') a set of corresponding reduced circuit invariants of the form (6). Note that the sets of repeating quantities discussed in Section 1 are precisely the basis sets.

Hence, if we are given a dimensional matrix \mathbf{D} , and we form the union over all basis sets of the sets of circuit invariants corresponding to sets of circuit sets associated with the current basis set, we are sure to include all invariants used in dimensional analysis based on \mathbf{D} .

Consider, for example, the main example in Section 2. By inspection of the dimensional matrix, we find ten basis sets, and there are two circuit invariants of the form (6) for each basis set. The basis sets and corresponding circuit invariants are shown below.

<i>Basis set</i>	<i>Circuit invariants</i>
$\{\triangle P/\ell, \rho, \mu\}$	$d^3 (\triangle P/\ell) \rho \mu^{-2}, u^3 (\triangle P/\ell)^{-1} \rho^2 \mu^{-1}$
$\{\triangle P/\ell, \rho, d\}$	$\mu^2 (\triangle P/\ell)^{-1} \rho^{-1} d^{-3}, u^2 (\triangle P/\ell)^{-1} \rho d^{-1}$
$\{\triangle P/\ell, \mu, d\}$	$\rho (\triangle P/\ell) \mu^{-2} d^3, u (\triangle P/\ell)^{-1} \mu d^{-2}$
$\{\triangle P/\ell, \rho, u\}$	$\mu (\triangle P/\ell) \rho^{-2} u^{-3}, d (\triangle P/\ell) \rho^{-1} u^{-2}$
$\{\triangle P/\ell, \mu, u\}$	$\rho^2 (\triangle P/\ell)^{-1} \mu^{-1} u^3, d^2 (\triangle P/\ell) \mu^{-1} u^{-1}$
$\{\triangle P/\ell, d, u\}$	$\rho (\triangle P/\ell)^{-1} d^{-1} u^2, \mu (\triangle P/\ell)^{-1} d^{-2} u$
$\{\rho, \mu, d\}$	$(\triangle P/\ell) \rho \mu^{-2} d^3, u \rho \mu^{-1} d$
$\{\rho, \mu, u\}$	$(\triangle P/\ell) \rho^{-2} \mu u^{-3}, d \rho \mu^{-1} u$
$\{\rho, d, u\}$	$(\triangle P/\ell) \rho^{-1} d u^{-2}, \mu \rho^{-1} d^{-1} u^{-1}$
$\{\mu, d, u\}$	$(\triangle P/\ell) \mu^{-1} d^2 u^{-1}, \rho \mu^{-1} d u$

The union of the ten sets of invariants is

$$\left\{ \frac{(\triangle P/\ell) \rho d^3}{\mu^2}, \frac{(\triangle P/\ell) \mu}{\rho^2 u^3}, \frac{(\triangle P/\ell) d}{\rho u^2}, \frac{(\triangle P/\ell) d^2}{\mu u}, \frac{\rho d u}{\mu} \right\}^{\pm 1}.$$

The set constructed in this way is a *sufficient set of minimal invariants* for \mathbf{D} , called the *unified basis* for \mathbf{D} , denoted $\mathcal{U}(\mathbf{D})$.

Alternatively, one can start with the set of circuit sets for \mathbf{D} . This set can be shown to be:

$$\{\{\triangle P/\ell, \rho, \mu, d\}, \{\triangle P/\ell, \rho, \mu, u\}, \{\triangle P/\ell, \rho, d, u\}, \{\triangle P/\ell, \mu, d, u\}, \{\rho, \mu, d, u\}\}.$$

Using \mathbf{D} , we find the set of pairs of circuit invariants corresponding to this set of circuit sets:

$$\left\{ \frac{(\triangle P/\ell) \rho d^3}{\mu^2}, \frac{(\triangle P/\ell) \mu}{\rho^2 u^3}, \frac{(\triangle P/\ell) d}{\rho u^2}, \frac{(\triangle P/\ell) d^2}{\mu u}, \frac{\rho d u}{\mu} \right\}^{\pm 1}.$$

We call a set of invariants obtained in this way the *circuit basis* for \mathbf{D} , denoted $\mathcal{C}(\mathbf{D})$. The Cocoa script in the Appendix implements an algorithm that can be used to calculate $\mathcal{C}(\mathbf{D})$ (and hence all circuit sets and circuit tuples as well).

The unified basis $\mathcal{U}(\mathbf{D})$ is obviously a subset of $\mathcal{C}(\mathbf{D})$. Equality does not always hold; for example, the dimensional matrix

$$X \quad \left\| \begin{array}{cc} q_1 & q_2 \\ 1 & -1 \end{array} \right\|$$

has the unified basis $\{q_1 q_2\}$ but the circuit basis $\{q_1 q_2\}^{\pm 1}$. In the case considered here $\mathcal{U}(\mathbf{D}) = \mathcal{C}(\mathbf{D})$, however, and this is the typical situation.

How big can $\mathcal{C}(\mathbf{D})$ be for a dimensional matrix of rank r with n columns? It can be shown that this number attains its maximum when the quantities in

all sets of r quantities are independent. Then all circuit sets contain exactly $r + 1$ quantities, the number of circuit sets is $\binom{n}{r+1}$, and the number of circuit invariants is $2\binom{n}{r+1}$. On the other hand, there are $n - r$ active minimal invariants in any relation of the form (2), and all these invariants are circuit invariants from different pairs of minimal invariants, so the number $\|\mathcal{C}(\mathbf{D})\|$ of *pairs* of circuit invariants satisfies the inequalities

$$n - r \leq \|\mathcal{C}(\mathbf{D})\| \leq \binom{n}{r+1}.$$

For the four dimensional matrices in Sections 2, 4 and 5, we have

<i>Section</i>	<i>n</i>	<i>r</i>	<i>n - r</i>	$\ \mathcal{C}(\mathbf{D})\ $	$\binom{n}{r+1}$
2 (1)	5	3	2	5	5
2 (2)	4	3	1	1	1
4	5	2	3	8	10
5	5	3	2	3	5

Remark 1

Let \sqsubseteq be a partial order on $\mathbb{Z}_0^n(\mathbf{D}) - \{\mathbf{0}\}$, the set of non-trivial integer solutions of (4), such that $(x_1, \dots, x_n) \sqsubseteq (y_1, \dots, y_n)$ if and only if $0 \leq x_i \leq y_i$ or $0 \geq x_i \geq y_i$ for $i = 1, \dots, n$. (Equivalently, $x_i y_i \geq 0$ and $|x_i| \leq |y_i|$ for $i = 1, \dots, n$.) The Graver basis for \mathbf{D} , denoted $\mathcal{G}(\mathbf{D})$, is the set of all minimal elements of $\mathbb{Z}_0^n(\mathbf{D}) - \{\mathbf{0}\}$ under this partial order.

Corresponding to the circuit basis of invariants $\mathcal{C}(\mathbf{D})$ as defined above, there is a *circuit basis of tuples* $\mathcal{C}_T(\mathbf{D})$ such that $(c_1, \dots, c_n) \in \mathcal{C}_T(\mathbf{D})$ if and only if $q_1^{c_1} \dots q_n^{c_n} \in \mathcal{C}(\mathbf{D})$. Every circuit tuple in $\mathcal{C}_T(\mathbf{D})$ is a clearly a minimal element of $\mathbb{Z}_0^n(\mathbf{D}) - \{\mathbf{0}\}$ under \sqsubseteq , so $\mathcal{C}_T(\mathbf{D}) \subset \mathcal{G}(\mathbf{D})$. Equality does not hold, however. For example, the matrix $\mathbf{D} = \begin{bmatrix} 1 & 2 & 1 \end{bmatrix}$ has the Graver basis

$$\{\pm(2, -1, 0), \pm(1, 0, -1), \pm(0, 1, -2), \pm(1, -1, 1)\},$$

but $(1, -1, 1)$ and $(-1, 1, -1)$ are not circuit tuples.

In a recent e-print [Ath13], sets of invariants for dimensional matrices corresponding to Graver bases for these matrices are presented. It is noted that “the Graver basis gives a full set of ... primitive invariants” for a dimensional matrix (p. 10). It has been shown here, however, that it suffices to consider circuit bases.

Remark 2

The terms ‘unified basis’ and ‘circuit basis’ are actually somewhat misleading, since the invariants in such bases are not independent in the usual sense. For example, $\rho^2 \mu^{-1} u^3 = \rho d^{-1} u^2 \cdot \rho u d \mu^{-1}$, and using such dependencies, the four

representations of the relation $\triangle P/\ell = \Psi(\rho, \mu, d, u)$ can be derived from each other by simple transformations. For example,

$$\begin{aligned} \frac{\triangle P/\ell}{\rho^2 \mu^{-1} u^3} = \phi_2(\rho u d \mu^{-1}) &\iff \frac{\triangle P/\ell}{\rho^2 \mu^{-1} u^3} \rho u d \mu^{-1} = \phi_2(\rho u d \mu^{-1}) \rho u d \mu^{-1} \\ &\iff \frac{\triangle P/\ell}{\rho d^{-1} u^2} = \phi_3(\rho u d \mu^{-1}). \end{aligned}$$

This manifest equivalence of representations, which is not surprising in view of the fact that they represent the same functional relation, has been invoked in a sophisticated justification of the tradition of limiting dimensional analysis to one relation and $n - r$ invariants. The basic argument is that since all representations of the functional relation are equivalent, it suffices to consider one of them, corresponding to one, arbitrarily chosen, basis set ([So01], p. 48). (A similar argument can be found already in Buckingham's original article on the Π theorem [Bu14], p. 362.) The examples in Section 2 should suffice to cast some doubt on this argument, however, and it is further weakened by examples of dimensional analysis presented in Sections 4 and 5.

In more abstract terms, we can regard the bijection $(c_1, \dots, c_n) \mapsto q_1^{c_1} \dots q_n^{c_n}$ as an isomorphism between the circuit basis of tuples $\mathcal{C}_T(\mathbf{D})$ and the circuit basis $\mathcal{C}(\mathbf{D})$. Since $\mathcal{C}_T(\mathbf{D})$ is basically an $(n - r)$ -dimensional solution space (over \mathbb{Z}), $\mathcal{C}(\mathbf{D})$ is an $(n - r)$ -dimensional space as well with operations defined by $q_1^{c_1} \dots q_n^{c_n} \cdot q_1^{d_1} \dots q_n^{d_n} = q_1^{c_1+d_1} \dots q_n^{c_n+d_n}$ and $(q_1^{c_1} \dots q_n^{c_n})^a = q_1^{ac_1} \dots q_n^{ac_n}$, and any choice of a set of repeating quantities is equivalent to a choice of a basis for $\mathcal{C}(\mathbf{D})$, where the $n - r$ basis elements correspond to the $n - r$ invariants in a representation of the given functional relation. While any two bases are equivalent in the sense that the elements of one can be expressed in terms of elements of the other, this does not mean that there cannot be any benefit from considering more than one basis. Similarly, the equivalence of representations in the sense indicated does not mean that only one representation of the functional relation should be considered.

4 Using some of the invariants

Consider quantities q_1, \dots, q_n , a dimensional matrix \mathbf{D} for these quantities, and a functional relation $q_1 = \Psi(q_2, \dots, q_n)$. The invariants, obtained by dimensional analysis, which appear in representations of this relation are all contained in $\mathcal{C}(\mathbf{D})$, but all invariants in $\mathcal{C}(\mathbf{D})$ are not used in the representations. This is basically because if a dependent quantity has been designated, we should disregard basis sets which include this quantity, because the column corresponding to the dependent quantity is always linearly dependent on the columns corresponding to the quantities in a basis set [Jo14]. Hence, only invariants associated with the non-disregarded basis sets will appear in the representations of the functional relation.

Consider, for example, the ten basis sets and associated circuit invariants in the preceding section. If $\triangle P/\ell$ is designated as the dependent quantity, meaning

that we assume a functional relation $\Delta P/\ell = \Psi(\rho, \mu, d, u)$ to hold, we should disregard the first six sets basis sets with associated sets of invariants. For the reader's convenience, the four last basis sets with associated sets of invariants are reproduced here:

<i>Basis set</i>	<i>Circuit invariants</i>
$\{\rho, \mu, d\}$	$(\Delta P/\ell) \rho \mu^{-2} d^3, u \rho \mu^{-1} d$
$\{\rho, \mu, u\}$	$(\Delta P/\ell) \rho^{-2} \mu u^{-3}, d \rho \mu^{-1} u$
$\{\rho, d, u\}$	$(\Delta P/\ell) \rho^{-1} d u^{-2}, \mu \rho^{-1} d^{-1} u^{-1}$
$\{\mu, d, u\}$	$(\Delta P/\ell) \mu^{-1} d^2 u^{-1}, \rho \mu^{-1} d u$

For each of these four basis sets there is a representation of $\Delta P/\ell = \Psi(\rho, \mu, d, u)$ involving the associated invariants; for $\{\rho, \mu, d\}$ we have $\Delta P/\ell = \frac{\mu^2}{\rho d^3} \phi_1\left(\frac{\rho u d}{\mu}\right)$, and so forth. These are the same formulas as given in Section 2, except that we obtain $\Delta P/\ell = \frac{\rho u^2}{d} \phi_3\left(\frac{\mu}{\rho u d}\right)$ instead of $\Delta P/\ell = \frac{\rho u^2}{d} \phi_3\left(\frac{\rho u d}{\mu}\right)$.

Thus, the basic reason why the circuit basis includes more invariants than are needed in the representations of any single functional relation is that a circuit basis can accommodate all possible functional relations among the quantities considered, or equivalently, all choices of a dependent quantity.

The following example, originally constructed by White and Lewalle, is adapted from [Whi03]. The displacement $S(t)$ of a falling body as a function of elapsed time t is given by the differential equation $S''(t) = g$, where g is the (local) constant of gravity. This equation has the solution $S(t) = S_0 + V_0 t + \frac{1}{2} g t^2$, where $S_0 = S(0)$ and $V_0 = S'(0)$. Thus, $S(t) = \Psi(S_0, V_0, g)(t)$. It is instructive to perform a dimensional analysis as if Ψ were an unknown function. The dimensional matrix is

$$\begin{array}{c} L \\ T \end{array} \left\| \begin{array}{ccccc} S(t) & S_0 & V_0 & g & t \\ 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 1 \end{array} \right\|.$$

This matrix has rank 2, so $n - r = 3$, but the circuit basis has 16 elements, namely $S(t) S_0^{-1}$, $S(t) V_0^{-2} g$, $S(t) V_0^{-1} g^{-1}$, $S(t) g^{-1} t^{-2}$, $S_0 V_0^{-2} g$, $S_0 V_0^{-1} t^{-1}$, $S_0 g^{-1} t^{-2}$, $V_0 g^{-1} t^{-1}$ and their inverses, and there are 9 basis sets: $\{S(t), V_0\}$, $\{S(t), g\}$, $\{S(t), t\}$, $\{S_0, V_0\}$, $\{S_0, g\}$, $\{S_0, t\}$, $\{V_0, g\}$, $\{V_0, t\}$, $\{g, t\}$.

As usual, we disregard basis sets containing the dependent quantity $S(t)$, and in this case we also disregard basis sets containing t because what we want to find out is how the functional relation $t \mapsto S(t)$ depends on the parameters S_0 , V_0 and g . The remaining basis sets and invariants are the following

<i>Basis set</i>	<i>Circuit invariants</i>
$\{S_0, V_0\}$	$S(t) S_0^{-1}, g S_0 V_0^{-2}, t V_0 S_0^{-1}$
$\{S_0, g\}$	$S(t) S_0^{-1}, V_0^2 S_0^{-1} g^{-1}, t^2 S_0^{-1} g$
$\{V_0, g\}$	$S(t) V_0^{-2} g, S_0 V_0^{-2} g, t V_0^{-1} g$

There are three corresponding representations of the functional relation $S(t) = \Psi(S_0, V_0, g)(t)$:

$$\begin{cases} S(t) = S_0 \Phi_1\left(\frac{q}{S_0^{-1}V_0^2}, \frac{t}{S_0V_0^{-1}}\right) & \text{(e)} \\ S(t) = S_0 \Phi_2\left(\frac{V_0}{\sqrt{S_0g}}, \frac{t}{\sqrt{S_0g^{-1}}}\right) & \text{(f)} \\ S(t) = V_0^2g^{-1} \Phi_3\left(\frac{S_0}{V_0^2g^{-1}}, \frac{t}{V_0g^{-1}}\right) & \text{(g)} \end{cases} .$$

It is pointed out in [Whi03] that while (e), (f) and (g) are representations of the same functional relation, and thus can be said to contain the same information, this information is presented in different ways, making it possible to draw different kinds of conclusions from the representations. For example, a plot of $S(t)/S_0$ as a function of $t/(S_0V_0^{-1})$ for different values of $g/(S_0^{-1}V_0^2)$ shows the effect of $g/(S_0^{-1}V_0^2)$ on the functional relation $t/(S_0V_0^{-1}) \mapsto S(t)/S_0$. Hence, this plot shows the effect of g on the functional relation $t \mapsto S(S_0, V_0, g)(t)$ for constant values of S_0 and V_0 . Plots corresponding to (f) and (g) similarly show the effect of V_0 and S_0 , respectively, on this functional relation.

Remark

This example is one of many showing that one can get more informative results from dimensional analysis by considering more than one functional relation and more than $n - r$ invariants. Yet, this fact is not reflected by the step-by-step algorithm for dimensional analysis presented in [Whi03]. This algorithm only describes how to obtain one relation involving $n - r$ invariants, so there is a tension between traditional principles and advanced practice in this textbook's exposition of dimensional analysis. It is argued here that this tension should be eliminated by modifying the principles.

5 Sets of representations of functional relations as equation systems

We have seen examples of how the existence of more than $n - r$ minimal invariants allows alternative representations of a relation of the form $q = \Psi(q_1, \dots, q_n)$ to be derived by means of dimensional analysis, but alternative representations are not only alternatives. These alternative representations can be regarded as an *equation system*, from which we can obtain more information about Ψ than is available from the representations considered separately. It may even be possible to determine Ψ (up to a multiplicative constant) from this equation system and available additional information. This will be shown by means of an example, adapted from [Bri22].

Let two bodies with mass m_1 and m_2 revolve around each other in circular orbits under influence of their mutual gravitational attraction. Let d denote

their distance and t the time of revolution. We want to derive a relation which shows how t depends on relevant parameters.

Preliminary considerations indicate that we should also include the universal gravitational constant G among the parameters, so that we obtain the dimensional matrix

$$\begin{array}{c} L \\ T \\ M \end{array} \quad \left\| \begin{array}{ccccc} t & d & m_1 & m_2 & G \\ 0 & 1 & 0 & 0 & 3 \\ 1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 1 & 1 & -1 \end{array} \right\|.$$

The circuit basis is

$$\left\{ \frac{m_1}{m_2}, \frac{t^2}{d^3 m_1^{-1} G^{-1}}, \frac{t^2}{d^3 m_2^{-1} G^{-1}} \right\}^{\pm 1},$$

the basis sets corresponding to the functional relation $t = \Psi(d, m_1, m_2, G)$ are $\{d, m_1, G\}$ and $\{d, m_2, G\}$, and the two sets of invariants associated with these two basis sets are $\{t^2 m_1 d^{-3} G, m_2 m_1^{-1}\}$ and $\{t^2 m_2 d^{-3} G, m_1 m_2^{-1}\}$, respectively, so the corresponding equation system is

$$\begin{cases} t^2 = \frac{d^3}{m_1 G} \Phi_1\left(\frac{m_2}{m_1}\right) & (\mathfrak{h}) \\ t^2 = \frac{d^3}{m_2 G} \Phi_2\left(\frac{m_1}{m_2}\right) & (\mathfrak{h}') \end{cases}.$$

This shows that t^2 is proportional to d^3 and inversely proportional to G , so we have derived Kepler's third law in a special case.

There is more information hidden in this equation system, however. In view of the symmetry between the two revolving bodies we may assume that $\Phi_1 = \Phi_2 = \Phi$. Multiplying (\mathfrak{h}) and (\mathfrak{h}') by $(m_1 G)/d^3$ and setting $x = m_1/m_2$, we obtain the functional equation

$$\Phi(1/x) = x \Phi(x),$$

which has solutions of the form

$$\Phi(x) = \frac{K}{1+x}.$$

Substituting this in (\mathfrak{h}) or (\mathfrak{h}') , we get

$$t^2 = \frac{K d^3}{G(m_1 + m_2)} \quad \text{or} \quad t = C \sqrt{\frac{d^3}{G(m_1 + m_2)}}.$$

There are more examples in [Jo14] showing that dimensional analysis can lead further than generally recognized if all relevant invariants are taken into account, which strengthens the conclusion that in general we should consider more than $n - r$ invariants and more than one relation of the form (2) in dimensional analysis.

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Appendix

Shown below is a simple CoCoA-5¹ script to help calculate the circuit basis.
(Edit to specify the dimensional matrix!)

```
"\n"; Use QQ;
-- Edit list of quantities:
vList := [ "S(t)", "S0", "V0", "g", "t" ];
-- Edit list of columns in dimensional matrix:
cList := [ [1,0], [1,0], [1,-1], [1,-2], [0,1] ];
n := len( vList );
r := rank( matrix( cList ) );
For i := 1 To r+1 Do
  lsv := subsets( vList, i );
  lsm := subsets( cList, i );
  m := len( lsm );
  For j := 1 To m Do
    sm := lsm[ j ];
    sm := Transposed( matrix( sm ) );
    If rank( sm ) = i-1 Then
      L := LinKerBasis( sm );
      If count( L[ 1 ], 0 ) = 0 Then
        lut := [ ];
        For k := 1 To len( lsv[ j ] ) Do
          append( ref lut, lsv[ j,k ] );
          append( ref lut, -L[ 1,k ] );
        EndFor;
        lut;
      EndIf;
    EndIf;
  EndFor;
EndFor;
```

Output from this script (corresponding to $2 \cdot 8$ circuit invariants):

```
["S(t)", -1, "S0", 1]
["S(t)", 1, "V0", -2, "g", 1]
["S(t)", -1, "V0", 1, "t", 1]
["S(t)", -1/2, "g", 1/2, "t", 1]
["S0", 1, "V0", -2, "g", 1]
["S0", -1, "V0", 1, "t", 1]
["S0", -1/2, "g", 1/2, "t", 1]
["V0", -1, "g", 1, "t", 1]
```

¹John Abbott, Anna Maria Bigatti, Giovanni Lagorio. CoCoA-5: a system for doing Computations in Commutative Algebra. Available at <http://cocoa.dima.unige.it>.